BORCHERDS' PROOF OF THE CONWAY-NORTON CONJECTURE

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ABSTRACT. We give a summary of R. Borcherds' solution (with some modifications) to the following part of the Conway-Norton conjectures: Given the Monster $\mathbb{M}$ and Frenkel-Lepowsky-Meurman's moonshine module $V^\natural$, prove the equality between the graded characters of the elements of $\mathbb{M}$ acting on $V^\natural$ (i.e., the McKay-Thompson series for $V^\natural$) and the modular functions provided by Conway and Norton. The equality is established using the homology of a certain subalgebra of the monster Lie algebra, and the Euler-Poincaré identity.

1. INTRODUCTION

In this paper we present a summary of R. Borcherds' proof of part of the Conway-Norton "monstrous moonshine" conjectures: the proof that the McKay-Thompson series of the Monster simple group acting on the known structure $V^\natural$ [FLM88] do indeed correspond to the Hauptmoduls presented in Conway and Norton [CN79]. Interested readers should certainly consult primary sources, a few of which are [B86], [FLM84], [FLM88], and [B92]. The simplification of the original proof, presented in this paper, can be found in [J98] and [JLW95]. See also Borcherds' survey articles about moonshine [B94] and [B98]. A brief overview of the historical development of the subject can be found in [FLM88]. What the reader will find in this paper is an outline of the proof itself, with references to particular results needed to establish the equality between the McKay-Thompson series for $V^\natural$ and the Laurent expansions of the relevant modular functions.

Given a group G and a $\mathbb{Z}$-graded G-module $V = \coprod_{n \in \mathbb{Z}} V_{[n]}$, with $\dim V_{[n]} < \infty$ for all $n \in \mathbb{Z}$ truncated so $V_{[k]} = 0$ for $k < N$ for some fixed $N \in \mathbb{Z}$, the McKay-Thompson series for $g \in G$ acting on V is defined to be the graded trace

$$T_g(q) = \sum_{n > N} \operatorname{tr}(g|V_{[n]}) q^n.$$

The moonshine conjectures of Conway and Norton in [CN79] include the conjecture that there should be an infinite-dimensional representation of the (not yet constructed) Fischer-Griess Monster simple group $\mathbb{M}$ such that the McKay-Thompson series T_g for $g \in \mathbb{M}$ acting on V have coefficients that are equal to the coefficients of the q-series expansions of certain modular functions. After the construction of $\mathbb{M}$ [Gr82], a "moonshine module" $V^\natural$ for the Monster simple group was constructed [FLM84], [FLM88] and many of its properties, including the determination of some

of its McKay-Thompson series, were proven. The critical example involves the modular function $j(\tau)$, a generator for the field of functions invariant under the action of $SL_2(\mathbb{Z})$ on the upper half plane. Let $q = e^{2\pi i \tau}$. Then the normalized q-series expansion denoted by $J(q) = j(q) - 744$ is one of the Hauptmoduls that occur in the moonshine correspondence [CN79]. Results of [FLM88] include the vertex operator algebra structure of $V^{\natural}$, with the conformal grading $V^{\natural} = \coprod_{i \geq 0} V_i^{\natural}$, as an $\mathbb{M}$-module, and the graded dimension correspondence $\dim V_i^{\natural} \leftrightarrow c(i-1)$ to the coefficients of the modular function $J(q) = \sum_{n \geq -1} c(n) q^n$. In other words, shifting the grading by defining $V_{[i-1]}^{\natural} = V_i^{\natural}$, we have $T_e(q) = J(q)$ as formal series, where $e \in \mathbb{M}$ is the identity element.

After the above results, the nontrivial problem of computing the rest of the McKay-Thompson series of Monster group elements acting on $V^{\natural}$ remained. Borcherds showed in [B92] that the McKay-Thompson series are the expected modular functions. The argument can be summarized as follows: Borcherds establishes a product formula

$$p(J(p) - J(q)) = \prod_{i=1,2,\ldots, j=-1,1,\ldots} (1 - p^i q^j)^{c(ij)}.$$

This formula is used in the proof, but first note that the formula leads to recursion formulas for the coefficients of the q-series expansion of $J(q)$, and hence to recursions for the dimensions of the homogeneous components of $V^{\natural}$. The approach of [B92] is to establish a product formula involving all of the McKay-Thompson series $T_g(q)$ of elements of $g \in \mathbb{M}$ acting on $V^{\natural}$, analogous to the above identity for $T_e(q) = J(q)$. The more general product formula in turn leads to a set of recursion formulas that determine the coefficients of the series, given a (large) set of initial data. For example, to determine $\sum \operatorname{tr}(g | V_{i+1}^{\natural}) q^i = \sum_{i \geq -1} c_g(i) q^i$ it is sufficient to compute four of the first five coefficients of $\sum \operatorname{tr}(g^k | V_{i+1}^{\natural}) q^i$, $k \in \mathbb{Z}$, $g \in \mathbb{M}$. One also determines that the Hauptmoduls listed in [CN79] satisfy the same recursion relations and initial data as the McKay-Thompson series for $V^{\natural}$.

The crucial product identity for the McKay-Thompson series is obtained by Borcherds from the Euler-Poincaré identity for the homology groups of a particular Lie algebra, the "monster" Lie algebra. This Lie algebra is constructed using the tensor product of the vertex operator algebra $V^{\natural}$ and a vertex algebra associated with a two-dimensional Lorentzian lattice. The monster Lie algebra , $\mathfrak{m}$, constructed in [B92] is an infinite-dimensional $\mathbb{Z} \times \mathbb{Z}$-graded Lie algebra. The Lie algebra $\mathfrak{m}$ is then shown to be a generalized Kac-Moody algebra, or a Borcherds algebra. The "No-ghost" theorem of string theory is used as a step in establishing an isomorphism between the $\mathbb{Z} \times \mathbb{Z}$-homogeneous components of $\mathfrak{m}$ and the weight spaces of $V^{\natural}$. The product formula for $p(J(p) - J(q))$ is interpreted as the denominator formula for the Lie algebra $\mathfrak{m}$ and used to determine the simple roots. The results pertaining to the homology groups of Lie algebras of [GL76] are then extended to include the class of Borcherds algebras and applied to a subalgebra $\mathfrak{n}^-$ of $\mathfrak{m}$ to obtain the desired family of identities. Of course, the recursions and initial data must also be established for the Hauptmoduls; see [Koi] and [Fer96].

In this paper we will actually discuss a modification of Borcherds proof, in which it is not necessary to generalize the homology results of [GL76]. We shall compute the homology groups as in [J98], [JLW95] with respect to a smaller subalgebra $\mathfrak{u}^- \subset \mathfrak{n}^-$. We shall use $\mathfrak{u}^-$ because it is a free Lie algebra, and therefore computing

the homology groups is straightforward. The Euler-Poincaré identity applied to the subalgebra $\mathfrak{u}^-$ and the trivial $\mathfrak{u}^-$-module $\mathbb{C}$ leads to recursions sufficient to establish the correspondence between the McKay-Thompson series for $V^\natural$ and the Hauptmoduls specified by Conway and Norton [CN79].

2. VERTEX OPERATOR ALGEBRAS.

We begin by recalling the definition of vertex operator algebra and vertex algebra. The following definition is a variant of Borcherds' original definition in [B86]. For a detailed discussion the reader can consult [FLM88], [FHL93], [DL93].

Definition 1. *A vertex operator algebra, $(V, Y, \mathbf{1}, \omega)$, consists of a vector space V, distinguished vectors called the vacuum vector $\mathbf{1}$ and the conformal vector ω, and a linear map $Y(\cdot, z) : V \to (End\ V)[[z, z^{-1}]]$ which is a generating function for operators v_n, i.e., for $v \in V$, $Y(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1}$, satisfying the following conditions:*

- **(V1):** $V = \coprod_{n \in \mathbb{Z}} V_n$*; for $v \in V_n$, $n = \mathrm{wt}(v)$*
- **(V2):** $\dim V_n < \infty$ *for $n \in \mathbb{Z}$*
- **(V3):** $V_n = 0$ *for n sufficiently small*
- **(V4):** *If $u, v \in V$ then $u_n v = 0$ for n sufficiently large*
- **(V5):** $Y(\mathbf{1}, z) = 1$
- **(V6):** $Y(v, z)\mathbf{1} \in V[[z]]$ *and $\lim_{z \to 0} Y(v, z)\mathbf{1} = v$, i.e., the creation property holds*
- **(V7):** *The following Jacobi identity holds:*

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y(u, z_1)Y(v, z_2) - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right) Y(v, z_2)Y(u, z_1)$$

(1)
$$= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) Y(Y(u, z_0)v, z_2).$$

The following conditions relating to the vector ω also hold;

- **(V8):** *The operators ω_n generate a Virasoro algebra i.e., if we let $L(n) = \omega_{n+1}$ for $n \in \mathbb{Z}$ then*

(2) $$[L(m), L(n)] = (m - n)L(m + n) + (1/12)(m^3 - m)\delta_{m+n,0}(rank V)$$

- **(V9):** *If $v \in V_n$ then $L(0)v = (\mathrm{wt}\ v)v = nv$*
- **(V10):** $\frac{d}{dz}Y(v, z) = Y(L(-1)v, z)$.

Definition 2. *A vertex algebra (with conformal vector) $(V, Y, \mathbf{1}, \omega)$ is a vector space V with all of the above properties except for **V2** and **V3**.*

For a vertex algebra V, and $v \in V$ with $\mathrm{wt}\ v = n$, let $(-z^{-2})^{L(0)}v = (-z^{-2})^n v$. This action extends linearly to all of V.

Definition 3. *A bilinear form on a vertex algebra V is invariant if for $v, u, w \in V$*

$$(Y(v, z)u, w) = (u, Y(e^{zL(1)}(-z^{-2})^{L(0)}v, z^{-1})w).$$

We note that an invariant form satisfies $(u, v) = 0$ unless $\mathrm{wt}(u) = \mathrm{wt}(v)$ for u, v homogeneous elements of V.

The tensor product of vertex algebras is also a vertex algebra [FHL93], [DL93]. Given two vertex algebras $(V, Y, \mathbf{1}_V, \omega_V)$ and $(W, Y, \mathbf{1}_W, \omega_W)$ the vacuum of $V \otimes W$

is $\mathbf{1}_V \otimes \mathbf{1}_W$ and the conformal vector ω is given by $\omega_V \otimes \mathbf{1}_W + \mathbf{1}_V \otimes \omega_W$. If the vertex algebras V and W both have invariant forms in the sense of Definition 3 then it follows from the definition of the tensor product that the form on $V \otimes W$ given by the product of the forms on V and W is also invariant.

One large and important class of vertex algebras are those associated with even lattices. Although the moonshine module $V^\natural$ is not a vertex operator algebra associated with a lattice (it is a far more complicated object), it is constructed using the vertex operator algebra associated to the Leech lattice. The vertex algebra used in the proof of the moonshine correspondence is the tensor product of the moonshine module and a vertex algebra associated with a two- dimensional Lorentzian lattice.

Given an even lattice L the vertex algebra V_L [B86] associated to the lattice has underlying vector space

$$V_L = S(\hat{\mathfrak{h}}_{\mathbb{Z}}^-) \otimes \mathbb{C}\{L\}.$$

(We are using the notation and constructions in [FLM88].) Here we take $\mathfrak{h} = L \otimes_{\mathbb{Z}} \mathbb{C}$, and $\hat{\mathfrak{h}}_{\mathbb{Z}}^-$ is the negative part of the Heisenberg algebra (with c central) defined by

$$\hat{\mathfrak{h}}_{\mathbb{Z}} = \coprod_{n \in \mathbb{Z}} \mathfrak{h} \otimes t^n \oplus \mathbb{C}c \subset \mathfrak{h} \otimes \mathbb{C}[[t]] \oplus \mathbb{C}c,$$

so that

$$\hat{\mathfrak{h}}_{\mathbb{Z}}^- = \coprod_{n < 0} \mathfrak{h} \otimes t^n.$$

The symmetric algebra on $\hat{\mathfrak{h}}_{\mathbb{Z}}^-$ is denoted $S(\hat{\mathfrak{h}}_{\mathbb{Z}}^-)$. Let $\hat{L}$ be a central extension of L by a group of order 2, i.e.,

$$1 \to \langle \kappa | \kappa^2 = 1 \rangle \to \hat{L} \overset{-}{\to} L \to 1,$$

with commutator map given by $\kappa^{\langle \alpha, \beta \rangle}$, $\alpha, \beta \in L$. Define $\mathbb{C}\{L\}$ to be the induced module $\mathrm{Ind}_{\langle \kappa \rangle}^{\hat{L}} \mathbb{C}$, where κ acts on $\mathbb{C}$ as multiplication by -1.

If $a \in \hat{L}$ denote by $\iota(a)$ the element $a \otimes 1 \in \mathbb{C}\{L\}$. We will use the notation $\alpha(n) = \alpha \otimes t^n \in S(\hat{\mathfrak{h}}_{\mathbb{Z}}^-)$. The vector space V_L is spanned by elements of the form:

$$(3) \qquad\qquad \alpha_1(-n_1)\alpha_2(-n_2)\dots\alpha_k(-n_k)\iota(a)$$

where $a \in \hat{L}, \alpha_i \in \mathfrak{h}$ and $n_i \in \mathbb{N}$. The space V_L, equipped with $Y(v, z)$ as defined in [FLM88] satisfies properties **V1** and **V4** $-$ **V10**, so is a vertex algebra with conformal vector ω.

A vertex algebra V_L constructed from an even lattice L automatically has an invariant bilinear form [B86], which can be defined using the contragredient module V_L' [FHL93].

3. Construction of the monster Lie algebra from the moonshine module.

The moonshine module $V^\natural$, a graded $\mathbb{M}$-module and vertex operator algebra, is constructed in [FLM88]. The following results describing the structure and properties of $V^\natural$ appear in Corollary 12.5.4 and Theorem 12.3.1 of [FLM88], part of which we restate here for the convenience of the reader. The invariance of the form in the sense of Definition 3 follows from the construction and results of [Li94]; see [J98]. Recall that $J(q)$ denotes the Laurent, or q-series, expansion of the modular function $j(\tau)$, normalized so that the coefficient of q^0 is zero.

Theorem 1.   (1) *The graded dimension of the moonshine module $V^\natural$ is $J(q)$.*
  (2) *$V^\natural$ is a vertex operator algebra of rank 24.*
  (3) *$\mathbb{M}$ acts in a natural way as automorphisms as of the vertex operator algebra $V^\natural$, i.e.,*
$$gY(v, z)g^{-1} = Y(gv, z)$$
  for $g \in \mathbb{M}, v \in V^\natural$
  (4) *There is an invariant positive definite hermitian form $(\cdot, \cdot)$ on $V^\natural$ which is also invariant under $\mathbb{M}$.*

In [B92] the monster Lie algebra is constructed using $V^\natural$ and the vertex algebra associated with a Lorentzian lattice as follows. Let $\Pi_{1,1} = \mathbb{Z} \oplus \mathbb{Z}$ be the rank two Lorentzian lattice with bilinear form $\langle \cdot, \cdot \rangle$ given by the matrix $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. The vertex algebra $V_{\Pi_{1,1}}$ has a conformal vector, and is given the structure of a trivial $\mathbb{M}$-module. Since $V_{\Pi_{1,1}}$ is a vertex algebra associated with an even lattice it has an invariant bilinear form, which we consider as $\mathbb{M}$-invariant under the trivial group action. Note that $V_{\Pi_{1,1}}$ is not a vertex operator algebra, because it does not satisfy conditions (**V2**) or (**V3**). For example, the weight of an element of the form (3) is $\sum_{i=1}^{k} n_i + \frac{1}{2} \langle a, a \rangle$, $n_i > 0 \in \mathbb{Z}$, $a \in \Pi_{1,1}$, which can be less than zero and arbitrarily large in absolute value.

Lemma 3.1. *The tensor product $V = V^\natural \otimes V_{\Pi_{1,1}}$ is a vertex algebra with conformal vector, and an invariant bilinear form, which is also $\mathbb{M}$-invariant.*

Given a vertex operator algebra V, or a vertex algebra V with conformal vector ω and therefore an action of the Virasoro algebra, let
$$P_i = \{v \in V | L(0)v = iv, L(n)v = 0 \text{ if } n > 0\}.$$
Thus P_i consists of the lowest weight vectors for the Virasoro algebra of conformal weight i. (P_1 is called the *physical space*). Let $u \in P_0$, then $\mathrm{wt}L(-1)u = \mathrm{wt}\omega + \mathrm{wt}u - 1 = 1$ and $L_{-1}P_0 \subset P_1$.

Lemma 3.2. *The space $P_1/L(-1)P_0$ is a Lie algebra with bracket given by*
$$[u + L(-1)P_0, v + L(-1)P_0] = u_0 v + L(-1)P_0.$$
For $u, v \in P_1$.

Proof. Let $u, v \in P_1$. By formula (8.8.7) of [FLM88] (see [B86])
$$(4) \qquad\qquad\qquad Y(u, z)v = e^{zL(-1)}Y(v, -z)u.$$
Taking coefficients of z^{-1} on both sides of (4) yields
$$u_0 v = -v_0 u + \sum_{k=1}^{\infty} (-1)^{k-1} L(-1)^k v_k u \in -v_0 u + L(-1)P_0.$$
Thus the bracket is anti-symmetric. Let $u, v, w \in P_1$ The Jacobi identity (**V7**) implies
$$u_0 v_0 w - v_0 u_0 w = (u_0 v)_0 w$$
$$(u_0(v_0 w)) - (v_0(u_0 w)) - ((u_0 v)_0 w) = 0.$$
Since we have shown anti-symmetry (modulo $L(-1)P_0$) the above is equivalent to the usual Lie algebra Jacobi identity for $[\cdot, \cdot]$ on $P_1/L(-1)P_0$. $\square$

We can now give the definition of Borcherds' monster Lie algebra. The tensor product $V = V^\natural \otimes V_{\Pi_{1,1}}$ is a vertex algebra with conformal vector, and invariant bilinear form. This form induces a bilinear form on the Lie algebra $P_1/L_{-1}P_0$. Note that if $u, v, w \in P_1$, then by invariance, and the fact that $L(1)u = 0$

$$(5) \quad (Y(u,z)v, w) = (v, Y(e^{zL(1)}(-z^{-2})^{L(0)}u, z^{-1})w)$$
$$= -(v, Y(u, z^{-1})z^{-2}w) = -(v, \sum_{n \in \mathbb{Z}} u_n z^{n-1}).$$

Taking the coefficient of z^{-1} we have for $u, v, w \in P_1/L(-1)P_0$

$$(u_0 v, w) = -(v, u_0 w),$$

and so the form on $P_1/L(1)P_0$ is invariant in the usual Lie algebra sense.

In addition to the weight grading, the vertex algebra $V^\natural \otimes V_{\Pi_{1,1}}$ is graded by the lattice $\Pi_{1,1}$. For u, v elements of degree $r, s \in \Pi_{1,1}$, the invariant form satisfies $(u, v) = 0$ unless $r = s$.

Let $N(\cdot, \cdot)$ denote the nullspace of the bilinear form on P_1 so $N(\cdot, \cdot) = \{u \in P_1 \mid (u, v) = 0 \quad \forall v \in P_1\}$. Since, for $u = L(-1)v, v \in P_0$, $w \in P_1$, it is immediate that $(L(-1)v, w) = (v, L(1)w) = 0$, we see $L(-1)P_0 \subset N(\cdot, \cdot)$. This in conjunction with Lemma 3.2 ensures the following is a Lie algebra.

Definition 4. *The monster Lie algebra $\mathfrak{m}$ is defined by*

$$\mathfrak{m} = P_1/N(\cdot, \cdot).$$

The monster Lie algebra is graded by the Lorentzian lattice $\Pi_{1,1}$ by construction. Elements of $\mathfrak{m}$ can be written as $\sum u \otimes ve^r$, where $u \in V^\natural$ and $ve^r = v\iota(e^r) \in V_{\Pi_{1,1}}$. Here, a section of the map $\hat{\Pi}_{1,1} \vec{\to} \Pi_{1,1}$ has been chosen so that $e^r \in \hat{\Pi}_{1,1}$ satisfies $\overline{e^r} = r \in \Pi_{1,1}$. There is a grading of $\mathfrak{m}$ by the lattice defined by $\deg(u \otimes ve^r) = r$. It follows from the construction that the Lie algebra $\mathfrak{m}$ has a Lie invariant bilinear form, whose radical is zero.

In order to establish the equality between the coefficients of the McKay-Thompson series for $V^\natural$ and the given Hauptmoduls, it is necessary to determine the dimensions of the components of $\mathfrak{m}$ of degrees $r \in \Pi_{1,1}$. Borcherds [B92] computes the dimensions by using Theorem 2 below, which uses the No-ghost theorem of string theory. For a proof of the No-ghost theorem see [GT72], [B92], or the appendix of [J98] for one written more algebraically.

Theorem 2. *Let V be a vertex operator algebra with the following properties:*

    i. *V has a symmetric invariant nondegenerate bilinear form.*
    ii. *The central element of the Virasoro algebra acts as multiplication by 24.*
    iii. *The weight grading of V is an $\mathbb{N}$-grading of V, i.e., $V = \coprod_{n=0}^{\infty} V_n$, and $\dim V_0 = 1$.*
    iv. *V is acted on by a group G preserving the above structure; in particular the form on V is G-invariant.*

Let $P_1 = \{u \in V \otimes V_{\Pi_{1,1}} | L(0)u = u, L(i)u = 0, i > 0\}$. The group G acts on $V \otimes V_{\Pi_{1,1}}$ via the trivial action on $V_{\Pi_{1,1}}$. Let P_1^r denote the subspace of P_1 of degree $r \in \Pi_{1,1}$. Then the quotient of P_1^r by the nullspace of its bilinear form is isomorphic as a G-module with G-invariant bilinear form to $V_{1-\langle r,r \rangle/2}$ if $r \neq 0$ and to $V_1 \oplus \mathbb{C}^2$ if $r = 0$.

Applying Theorem 2 to $V = V^\natural \otimes V_{\Pi_{1,1}}$ we see that the monster Lie algebra has $(m,n) \in \mathbb{Z} \times \mathbb{Z}$ homogeneous subspaces isomorphic to the weight spaces $V^\natural_{mn+1}$ when $(m,n) \neq (0,0)$, that is, $\mathfrak{m}_{(m,n)} = V^\natural_{[mn]}$. We have shown:

$$\mathfrak{n}^+ = \coprod_{m>0, n \geq -1} \mathfrak{m}_{(m,n)},$$

with

$$\mathfrak{m}_{(m,n)} \simeq V^\natural_{mn+1}$$

and similarly for $\mathfrak{n}^-$.

4. The structure of the monster Lie algebra

A crucial step in [B92] is to identify $\mathfrak{m} = P_1/N(\cdot,\cdot)$ with a Lie algebra given by a generalization of a Cartan matrix. This allows one to be able to compute the homology groups of the trivial module $\mathbb{C}$ with respect to an appropriate subalgebra of $\mathfrak{m}$, as can be done for symmetrizable Kac-Moody Lie algebras [GL76], [Liu92].

The Lie algebra $\mathfrak{g}(A)$ associated to a symmetrizable matrix A is introduced in [K90] and [Mo67], but the systematic study of the case where A satisfies conditions **B1-B3** below was carried out by Borcherds. Borcherds algebras have many properties in common with symmetrizable Kac-Moody algebras such as an invariant bilinear form and a root lattice grading. One notable difference is that there may be simple imaginary roots in the root lattice. This is a desirable property in the monster case $\mathfrak{m}$ because we wish to associate a root grading to the hyperbolic $\mathbb{Z} \times \mathbb{Z}$-grading inherited from $V_{\Pi_{1,1}}$.

We review the construction of the Borcherds algebra $\mathfrak{g}(A)$ of [B88]. Let I be a (finite or) countable index set and let $A = (a_{ij})_{i,j \in I}$ be a matrix with entries in $\mathbb{C}$, satisfying the following conditions:

(B1): A is symmetric.
(B2): If $i \neq j$ $(i,j \in I)$, then $a_{ij} \leq 0$.
(B3): If $a_{ii} > 0$ $(i \in I)$, then $2a_{ij}/a_{ii} \in \mathbb{Z}$ for all $j \in I$.

Let $\mathfrak{g}'(A)$ be the Lie algebra with generators h_i, e_i, f_i, $i \in I$, and the following defining relations: For all $i,j,k \in I$,

$$[h_i, h_j] = 0, [e_i, f_j] - \delta_{ij} h_i = 0,$$
$$[h_i, e_k] - a_{ik} e_k = 0, [h_i, f_k] + a_{ik} f_k = 0$$

and Serre relations

$$(\operatorname{ad} e_i)^{-2a_{ij}/a_{ii}+1} e_j = 0, (\operatorname{ad} f_i)^{-2a_{ij}/a_{ii}+1} f_j = 0$$

for all $i \neq j$ with $a_{ii} > 0$, and finally

$$[e_i, e_j] = 0, [f_i, f_j] = 0$$

whenever $a_{ij} = 0$.

Let $\mathfrak{h} = \sum_{i \in I} \mathbb{C} h_i$, $\mathfrak{n}^\pm$ the subalgebra generated by the elements e_i (resp. the f_i) for $i \in I = \langle e_i \rangle$. As in the Kac-Moody case, the simple roots $\alpha_i \in (\mathfrak{h})^*$ are defined to satisfy $(\alpha_i, \alpha_j) = a_{ij}$. Also as in the Kac-Moody case, we may have linearly dependent simple roots α_i and we extend the Lie algebra as in [GL76] and [Le79] by an appropriate abelian Lie algebra $\mathfrak{d}$ of degree derivations, chosen so that the simple roots are linearly independent in $(\mathfrak{h} \ltimes \mathfrak{d})^*$.

Definition 5. *The Lie algebra* $\mathfrak{g}(A) = g'(A) \ltimes \mathfrak{d}$ *is the Borcherds or generalized Kac-Moody (Lie) algebra associated to the matrix A. Any Lie algebra of the form $\mathfrak{g}(A)/\mathfrak{c}$ where $\mathfrak{c}$ is a central ideal is also called a Borcherds algebra.*

Versions of the following theorem appear in [B88] and [B95], see also [J98]. This theorem allows us to recognize a Lie algebra associated to a matrix A satisfying **B1 − B3**.

Theorem 3. *Let $\mathfrak{g}$ be a Lie algebra satisfying the following conditions:*
   (1) *$\mathfrak{g}$ can be $\mathbb{Z}$-graded as $\coprod_{i \in \mathbb{Z}} \mathfrak{g}_i$, $\mathfrak{g}_i$ is finite dimensional if $i \neq 0$, and $\mathfrak{g}$ is diagonalizable with respect to $\mathfrak{g}_0$.*
   (2) *$\mathfrak{g}$ has an involution η which maps $\mathfrak{g}_i$ onto $\mathfrak{g}_{-i}$ and acts as -1 on noncentral elements of $\mathfrak{g}_0$, in particular, $\mathfrak{g}_0$ is abelian.*
   (3) *$\mathfrak{g}$ has a Lie algebra-invariant bilinear form $(\cdot, \cdot)$, invariant under η, such that $\mathfrak{g}_i$ and $\mathfrak{g}_j$ are orthogonal if $i \neq -j$, and such that the form $(\cdot, \cdot)_0$, defined by $(x, y)_0 = -(x, \eta(y))$ for $x, y \in \mathfrak{g}$, is positive definite on $\mathfrak{g}_m$ if $m \neq 0$.*
   (4) *$\mathfrak{g}_0 \subset [\mathfrak{g}, \mathfrak{g}]$.*
Then there is a central extension $\hat{\mathfrak{g}}$ of a Borcherds algebra and a homomorphism, π, from $\hat{\mathfrak{g}}$ onto $\mathfrak{g}$, such that the kernel of π is in the center of $\hat{\mathfrak{g}}$.

The theorem is proven by inductively constructing a set of generators of $\mathfrak{g}_n$, $n \in \mathbb{Z}$, consisting of $\mathfrak{g}_0$ weight vectors, using the form $(x, y)_0$. Proofs can be found in [B91], see [J98] for the theorem stated exactly as above. An alternative characterization of Borcherds algebra can be found in [B92].

Theorem 4. *The Lie algebra $\mathfrak{m} = P_1/N(\cdot, \cdot)$ is a Borcherds algebra.*

Proof. The abelian subalgebra $\mathfrak{m}_{(0,0)}$ is spanned by elements of the form $1 \otimes \alpha(-1)\iota(1)$ where $\alpha \in \Pi_{1,1} \otimes_{\mathbb{Z}} \mathbb{C}$. Note that $\mathfrak{m}_{(0,0)}$ is two-dimensional. In order to apply Theorem 3, grade $\sum_{(m,n) \in \Pi_{1,1}} m_{(m,n)}$ by $i = 2m + n \in \mathbb{Z}$. With this grading, $\mathfrak{m}$ satisfies condition (i) of Theorem 3.

There is an involution η is on the vertex algebra $V_{\Pi_{1,1}}$, determined by $\eta(\alpha) = -\alpha$ for $\alpha \in \Pi_{1,1}$. Extend the involution to $V^{\natural} \otimes V_{\Pi_{1,1}}$ by taking $\eta(\sum u \otimes v) = \sum(u \otimes \eta v)$ for $u \in V^{\natural}, v \in V_{\Pi_{1,1}}$. The invariant form given by Lemma 3.1 is the required non-degenerate invariant bilinear form, satisfying condition (iii) in Theorem 3. Let $a = e^{(1,1)}$, $b = e^{(1,-1)}$. Condition (iv) follows from the fact that $\mathfrak{m}_{(0,0)}$ is two-dimensional and that the elements $[u \otimes \iota(a), v \otimes \iota(a^{-1})]$ and $[\iota(b), \iota(b^{-1})]$ for $u, v \in V_2^{\natural}$ are two linearly independent vectors in $\mathfrak{m}_{(0,0)}$. Thus the Lie algebra $\mathfrak{m}$ is the homomorphic image of some Borcherds algebra $\mathfrak{g}(A)$ associated to a matrix. $\square$

By computing the action of $a \in \mathfrak{g}_0 = \mathfrak{m}_{(0,0)}$ on $v \in \mathfrak{m}_r$, $r \in \Pi_{1,1}$ one obtains $[a, v] = \langle \alpha, r \rangle v$ and Borcherds identifies the elements of $\Pi_{1,1}$ with the root lattice of $\mathfrak{m}$. The following is Theorem 7.2 of [B92].

Theorem 5. *The simple roots of the monster Lie algebra $\mathfrak{m}$ are the vectors $(1, n)$, $n = -1$ or $n > 0$, each with multiplicity $c(n)$.*

This theorem is proven [B92] by identifying the product formula

$$p(J(p) - J(q)) = \prod_{i=1,2,\ldots, j=-1,1,\ldots} (1 - p^i q^j)^{c(ij)}$$

with the denominator identity for the Borcherds algebra $\mathfrak{m}$.

Since, by definition of $\mathfrak{m}$ the radical of the invariant form on $\mathfrak{m}$ is zero, the kernel of the homomorphism in Theorem 3 is in the center of $\mathfrak{g}(A)$. We construct a symmetric matrix B, determined by the root lattice $\Pi_{1,1}$ and the multiplicities given by Theorem 2. We have the Lie algebra $\mathfrak{m}$ is isomorphic to $\mathfrak{g}(B)/\mathfrak{c}$, where $\mathfrak{g}(B)$ is the Borcherds algebra associated to the following matrix B and $\mathfrak{c}$ is the full center of $\mathfrak{g}(B)$:

$$
B = \left(\begin{array}{c|ccc|ccc|c}
2 & 0 & \cdots & 0 & -1 & \cdots & -1 & \cdots \\ \hline
0 & -2 & \cdots & -2 & -3 & \cdots & -3 & \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \cdots \\
0 & -2 & \cdots & -2 & -3 & \cdots & -3 & \\ \hline
-1 & -3 & \cdots & -3 & -4 & \cdots & -4 & \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \cdots \\
-1 & -3 & \cdots & -3 & -4 & \cdots & -4 & \\ \hline
\vdots & & \vdots & & & \vdots & &
\end{array}\right).
$$

In this summary of Borcherds' proof of part of the moonshine conjectures we are able to bypass the part of the argument of [B92] that requires a more extensive development of the theory of Borcherds algebras including generalizing the results of [GL76]. Instead, we will use Theorem 6 below, proven in [J98]. Given a vector space U, let $L(U)$ denote the free Lie algebra generated by a basis of U. Let $J \subset I$ be the set $\{i \in I | a_{ii} > 0\}$. Note that the matrix $(a_{ij})_{i,j \in J}$ is a generalized Cartan matrix. Let $\mathfrak{g}_J$ be the Kac-Moody algebra associated to this matrix. Then $\mathfrak{g}_J = \mathfrak{n}_J^+ \oplus \mathfrak{h}_J \oplus \mathfrak{n}_J^-$, and $\mathfrak{g}_J$ is isomorphic to the subalgebra of $\mathfrak{g}(A)$ generated by $\{e_i, f_i\}$ with $i \in J$.

Theorem 6. *Let A be a matrix satisfying conditions* **B1-B3**. *Let J and $\mathfrak{g}_J$ be as above. Assume that if $i, j \in I\backslash J$ and $i \neq j$ then $a_{ij} < 0$. Then*

$$\mathfrak{g}(A) = \mathfrak{u}^+ \oplus (\mathfrak{g}_J + \mathfrak{h}) \oplus \mathfrak{u}^-,$$

where $\mathfrak{u}^- = L(\coprod_{j \in I\backslash J} U(\mathfrak{n}_J^-) \cdot f_j)$ and $\mathfrak{u}^+ = L(\coprod_{j \in I\backslash J} U(\mathfrak{n}_J^+) \cdot e_j)$. The $U(\mathfrak{n}_J^-) \cdot f_j$ for $j \in I\backslash J$ are integrable highest weight $\mathfrak{g}_J$-modules, and the $U(\mathfrak{n}_J^+) \cdot e_j$ are integrable lowest weight $\mathfrak{g}_J$-modules.

Note that the conditions on the a_{ij} given in the theorem are equivalent to the statement that the Lie algebra has no mutually orthogonal imaginary simple roots. This is the case for the monster Lie algebra $\mathfrak{m}$.

The structure of $\mathfrak{m}$ can now be summarized. There are natural isomorphisms

$$\mathfrak{m}_{(m,n)} \cong V_{mn+1}^{\natural} \text{ as an } M\text{-module for } (m,n) \neq (0,0),$$

$$\mathfrak{m}_{(0,0)} \cong \mathbb{C} \oplus \mathbb{C}, \text{ a trivial } M\text{-module.}$$

It follows from the definition of $\mathfrak{m}$ that

$$\mathfrak{m}_{(-1,1)} \oplus \mathfrak{m}_{(0,0)} \oplus \mathfrak{m}_{(1,-1)} \cong \mathfrak{gl}_2.$$

Applying Theorem 6 to the above realization of $\mathfrak{m}$ by generators and relations gives

$$\mathfrak{m} = \mathfrak{u}^+ \oplus \mathfrak{gl}_2 \oplus \mathfrak{u}^-,$$

with $\mathfrak{u}^- = L(U)$ and $\mathfrak{u}^+ = L(U')$. Where $L(U), L(U')$ are free Lie algebras over vector spaces that are direct sums of $\mathfrak{gl}_2$-modules.

$$U = \coprod_{i>0} W_i \otimes V_{i+1}^\natural \text{ and } U' = \coprod_{i>0} W_i' \otimes V_{i+1}^\natural.$$

For $i > 0$, $V_{i+1}^\natural$ is (as usual) the weight $i + 1$ component of $V^\natural$, W_i denotes the (unique up to isomorphism) irreducible highest weight $\mathfrak{gl}_2$-module of dimension i on which z acts as $i + 1$ and W_i', $i > 0$, denotes the irreducible lowest weight module.

5. The homology computation and recursion formulas

We are now ready to establish the recursion relations for the coefficients of the McKay-Thompson series $\sum_{i>0} \mathrm{Tr}(g|V_{i+1}^\natural)q^i = \sum_{i\in\mathbb{Z}} c_g(i)q^i$. What follows is a summary of what has appeared in [JLW95]. See [Ka94] and [KK95] for similar computations.

To compute the homology of the free Lie algebra $L(U)$ for a vector space U, with coefficients in the trivial module (as in [CE56]), consider the following exact sequence is a $U(L(U)) = T(U)$-free resolution of the trivial module:

$$0 \to T(U) \otimes U \overset{\mu}{\to} T(U) \overset{\epsilon}{\to} \mathbb{C} \to 0$$

where μ is the multiplication map and ϵ is the augmentation map. One obtains :

$$H_0(L(U), \mathbb{C}) = \mathbb{C}$$
$$H_1(L(U), \mathbb{C}) = U \cong L(U)/[L(U), L(U)]$$
$$H_n(L(U), \mathbb{C}) = 0 \text{ for } n \geq 2.$$

Let p, q and t be commuting formal variables. The variables p^{-1} and q^{-1} will be used to index the $\mathbb{Z} \oplus \mathbb{Z}$-grading of our vector spaces. All of the $\mathbb{M}$-modules we encounter are finite-dimensionally $\mathbb{Z} \oplus \mathbb{Z}$-graded with grading suitably truncated and will be identified with formal series in $R(\mathbb{M})[[p,q]]$. Definitions and results from [Kn73] about the λ-ring $R(\mathbb{M})$ of finite-dimensional representations of $\mathbb{M}$ are applicable to formal series in $R(\mathbb{M})[[p,q]]$. We summarize the results of, for example, [Kn73] that we use below.

The representation ring $R(\mathbb{M})$ is a λ-ring [Kn73] with the λ operation given by exterior powers, so $\lambda^i V = \bigwedge^i V$ for $V \in R(\mathbb{M})$.

In the following discussion we let $W, V \in R(\mathbb{M})$. The operation $\bigwedge^i$ satisfies

$$\bigwedge{}^i(W \oplus V) = \sum_{n=0}^{i} \bigwedge{}^n(W) \otimes \bigwedge{}^{i-n}(V).$$

Define

$$\bigwedge{}_t(W) = \bigwedge{}^0(W) + \bigwedge{}^1(W)t + \bigwedge{}^2(W)t^2 + \cdots.$$

Then

(6) $$\bigwedge{}_t(V \oplus W) = \bigwedge{}_t(V) \cdot \bigwedge{}_t(W).$$

The Adams operations $\Psi^k : R(\mathbb{M}) \to R(\mathbb{M})$ are defined for $W \in R(\mathbb{M})$ by:

(7) $$\frac{d}{dt} \log \bigwedge{}_t(W) = \sum_{n\geq 0} (-1)^n \Psi^{n+1}(W)t^n.$$

For a class function $f : \mathbb{M} \to \mathbb{C}$, define

$$(\Psi^k f)(g) = f(g^k).$$

for all $g \in \mathbb{M}$.

Now let W be a finite-dimensionally $\mathbb{Z} \oplus \mathbb{Z}$- graded representation of $\mathbb{M}$ such that $W_{(\gamma_1,\gamma_2)} = 0$ for $\gamma_1, \gamma_2 > 0$. We shall write $W = \sum_{(\gamma_1,\gamma_2)\in\mathbb{N}^2} W_{(-\gamma_1,-\gamma_2)}p^{\gamma_1}q^{\gamma_2}$, identifying the graded space and formal series. We extend the definition of Ψ^k to formal series $W \in R(\mathbb{M})[[p,q]]$ by defining $\Psi^k(p) = p^k$, $\Psi^k(q) = q^k$ and in general,

$$\Psi^k\Big(\sum_{(\gamma_1,\gamma_2)\in\mathbb{N}^2} W_{(-\gamma_1,-\gamma_2)}p^{\gamma_1}q^{\gamma_2}\Big) = \sum_{(\gamma_1,\gamma_2)\in\mathbb{N}^2} \Psi^k(W_{(-\gamma_1,-\gamma_2)})p^{k\gamma_1}q^{k\gamma_2}.$$

Recall the structure of $\mathfrak{u}^-$ and $U = H_1(\mathfrak{u}^-)$ as $\mathbb{Z} \oplus \mathbb{Z}$-graded M-modules. We index the grading by p^{-1} and q^{-1}; then write $\mathfrak{u}^-$ and $U = H_1(\mathfrak{u}^-)$ as elements of $R[M][[p,q]]$:

$$\text{(8)} \qquad\qquad \mathfrak{u}^- = \sum_{(m,n)} V^\natural_{mn+1}p^m q^n$$

and

$$\text{(9)} \qquad\qquad U = \sum_{(m,n)} V^\natural_{m+n}p^m q^n,$$

where here and below the sums are over all pairs (m,n) such that $m, n > 0$.

Define

$$H_t(\mathfrak{u}^-) = \sum_{i=0}^{\infty} H_i(\mathfrak{u}^-)t^i$$

and let $H(\mathfrak{u}^-)$ denote the alternating sum $H_t(\mathfrak{u}^-)|_{t=-1}$. Recall the Euler-Poincaré identity:

$$\text{(10)} \qquad\qquad \textstyle\bigwedge_{-1}(\mathfrak{u}^-) = H(\mathfrak{u}^-).$$

Taking log of both sides of (10) results in the formal power series identity in $R(M)[[p,q]] \otimes \mathbb{Q}$:

$$\text{(11)} \qquad\qquad \log\textstyle\bigwedge_{-1}(\mathfrak{u}^-) = \log H(\mathfrak{u}^-),$$

where we have

$$\log H(\mathfrak{u}^-) = \log(1 - H_1(\mathfrak{u}^-)) = -\sum_{n=1}^{\infty}\frac{1}{n}H_1(\mathfrak{u}^-)^n.$$

Formally integrating (7), with $W = \mathfrak{u}^-$, gives

$$\log\textstyle\bigwedge_t(\mathfrak{u}^-) = -\sum_{n\geq 0}\Psi^{n+1}(\mathfrak{u}^-)\frac{(-t)^{n+1}}{n+1}.$$

Then setting $t = -1$ gives:

$$-\log\textstyle\bigwedge_{-1}(\mathfrak{u}^-) = \sum_{k=1}^{\infty}\frac{1}{k}\Psi^k(\mathfrak{u}^-).$$

Since $H_1(\mathfrak{u}^-) = U$, equation (11) gives

$$\text{(12)} \qquad \sum_{k=1}^{\infty}\frac{1}{k}\Psi^k\Big(\sum_{(m,n)} V^\natural_{mn+1}p^m q^n\Big) = \sum_{k=1}^{\infty}\frac{1}{k}\Big(\sum_{(m,n)} V^\natural_{m+n}p^m q^n\Big)^k$$

We say $k|(i,j)$ if $k(m,n) = (i,j)$ for some $(m,n) \in \mathbb{Z} \oplus \mathbb{Z}$. For $(i,j) \in \mathbb{Z}_+ \oplus \mathbb{Z}_+$ we define

$$P(i,j) = \{a = (a_{rs})_{r,s \in \mathbb{Z}_+} \mid a_{rs} \in \mathbb{N}, \sum_{(r,s) \in \mathbb{Z}_+ \oplus \mathbb{Z}_+} a_{rs}(r,s) = (i,j)\}.$$

We will use the notation $|a| = \sum a_{rs}$, $a! = \prod a_{rs}!$. Expanding both sides of equation (12)

$$(13) \quad \sum_{(i,j)} \sum_{k|(i,j)} \frac{1}{k} \Psi^k(V^\natural_{ij/k^2+1}) p^i q^j = \sum_{(i,j)} \sum_{a \in P(i,j)} \frac{(|a|-1)!}{a!} \prod_{r,s \in \mathbb{Z}_+} (V^\natural_{r+s})^{a_{rs}} p^i q^j.$$

Then taking the trace of an element $g \in \mathbb{M}$ on both sides of the identity (13)

$$\sum_{(i,j)} \sum_{k|(i,j)} \frac{1}{k} \Psi^k(c_g(ij/k^2)) p^i q^j$$

$$= \sum_{(i,j)} \sum_{a \in P(i,j)} \frac{(|a|-1)!}{a!} \prod_{r,s \in \mathbb{Z}_+} c_g(r+s-1)^{a_{rs}} p^i q^j.$$

Equating the coefficients of $p^i q^j$ and applying Möbius inversion yields the recursion formulas:

$$(14) \quad c_g(ij) = \sum_{\substack{k > 0 \\ k(m,n)=(i,j)}} \frac{1}{k} \mu(k) \Big(\sum_{a \in P(m,n)} \frac{(|a|-1)!}{a!} \prod_{r,s \in \mathbb{Z}_+} c_{g^k}(r+s-1)^{a_{rs}} \Big).$$

The coefficients of any replicable function are determined by the first 23 coefficients [CN95], but in the case of the above McKay-Thompson series we can use a smaller set of coefficients.

An examination of the formula (14) shows that $c_g(n)$ is determined by expressions of lower level except when $n = 1, 2, 3, 5$. Thus the values of the $c_g(n)$ are determined by the $c_h(1)$, $c_h(2)$, $c_h(3)$, $c_h(5)$, $h \in \mathbb{M}$, and the above recursions.

As in [B92], we conclude that since both the McKay-Thompson series for $V^\natural$ and the modular functions of [CN79] satisfy (14), all that is necessary to prove that these functions are the same is to check the initial data listed above. For the modular functions, see [Koi] and [Fer96]. For the relevant initial data about the graded traces of the actions of the elements of the Monster on $V^\natural$, the main theorem of [FLM88] is needed, as used in [B92].

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